Fields of Quantum Reference Frames based on Different Representations of Rational Numbers as States of Qubit Strings

Paul Benioff

Physics Division, Argonne National Laboratory

Argonne, IL 60439

e-mail: pbenioff@anl.gov

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Abstract

In this paper fields of quantum reference frames based on gauge transformations of rational string states are described in a way that, hopefully, makes them more understandable than their description in an earlier paper. The approach taken here is based on three main points: (1) There are a large number of different quantum theory representations of natural numbers, integers, and rational numbers as states of qubit strings. (2) For each representation, Cauchy sequences of rational string states give a representation of the real (and complex) numbers. A reference frame is associated to each representation. (3) Each frame contains a representation of all mathematical and physical theories that have the real and complex numbers as a scalar base for the theories. These points and other aspects of the resulting fields are then discussed and justified in some detail. Also two different methods of relating the frame field to physics are discussed.

1 Introduction

In other work [1] two dimensional fields of quantum reference frames were described that were based on different quantum theory representations of the real numbers. Because the description of the fields does not seem to be understood, it is worthwhile to approach a description in a way that will, hopefully, help to make the fields better understood. This the goal of this contribution to the third Feynman conference proceedings.

The approach taken here is based on three main points:

- There are a large number of different quantum theory representations of natural numbers, integers, and rational numbers as states of qubit strings. These arise from gauge transformations of the qubit string states.

- For each representation, Cauchy sequences of rational string states give a representation of the real (and complex) numbers. A reference frame is associated to each representation.

- Each frame contains a representation of all mathematical and physical theories. Each of these is a mathematical structure that is based on the real and complex number representation base of the frame.

This approach is summarized in the next section with more details given in the following sections.

2 Summary Discussion of the Three Points

As is well known, the large amount of work and interest in quantum computing and quantum information theory is based on quantum mechanical representations of numbers as states of strings of qubits and linear superpositions of these states. The numbers represented by these states are usually the nonnegative integers or the natural numbers. Examples of integer representations are states of the form $|\gamma, s\rangle$ and their linear superpositions $\psi = \sum_{s,\gamma} c(\gamma, s)|\gamma, s\rangle$. Here $|\gamma, s\rangle = |\gamma, 0\rangle \otimes_{j=1}^{j=n} |s(j), j\rangle$ is a state of a string of $n+1$ qubits where the qubit at location 0 denotes the sign (as $\gamma = +, -$) and $s : [1, \cdots, n] \rightarrow \{0, 1\}$ is a $0-1$ valued function on the integer interval $[1, n]$ where $s(n) = 1$. This last condition removes the redundancy of leading $0s$.

This description can be extended to give quantum mechanical representations of rational numbers as states of qubit strings [2] and of real and complex numbers as Cauchy sequences of rational number states of qubit strings [3]. As will be seen, string rational numbers can be represented by qubit string states $|\gamma, s\rangle$ where s is a $0-1$ valued function on an integer interval $[l, u]$ with $l \leq 0$ and $u \geq 0$ and the sign qubit γ, at position 0.

A basic point to note is that there are a great many different representations of rational numbers (and of integers and natural numbers) as states of qubit strings. Besides the arbitrariness of the location of the qubit string on an integer lattice there is the arbitrariness of the choice of quantization axis for each qubit in the string. This latter arbitrariness is equivalent to a gauge freedom for the choice of which system states correspond to the qubit states $|0\rangle$ and $|1\rangle$.

This arbitrariness of gauge choice for each qubit is discussed in Section 4 in terms of global and local gauge transformations of rational string states of qubit strings. A different representation of string rational numbers as states of qubit strings is associated with each gauge transformation.

As will be seen in the next section, there is a quantum representation of real numbers associated with each representation of the rational string states. It is clear that there are a large number of different representations of real numbers as each representation is associated with a different gauge gauge representation of the rational string states.

The gauge freedom in the choice of quantization axis for each qubit plays an important role in quantum cryptography and the transfer of quantum infor-

mation between a sender and receiver. The choice of axis is often referred to a reference frame chosen by sender and receiver for transformation and receipt of quantum information [4, 5, 6, 7].

Here this idea of reference frames is taken over in that a reference frame F_{R_U} is associated with each quantum representation R_U of real numbers. Since each real number representation is associated with a gauge transformation U, one can also associate frames directly with gauge transformations, as in $U \rightarrow F_U$ instead of F_{R_U}.

It should be noted that complex numbers are also included in this description since they can be represented as an ordered pair of real numbers. Or they can be built up directly from complex rational string states. From now on real numbers and their representations will be assumed to also include complex numbers and their representations.

An important point for this paper is that any physical or mathematical theory, for which the real numbers form a base or scalar set, has a representation in each frame as a mathematical structure based on the real number representation in the frame. Since this is the case for all physical theories considered to date, it follows that they all have representations in each frame as mathematical structures based on the real number representation in the frame. It follows that theories such as quantum mechanics, quantum field theory. QED, QCD, string theory, and special and general relativity all have representations in each frame as mathematical structures based on the real number representation associated with the frame. It is also the case that, if the space time manifold is considered to be a $4-tuple$ of the real numbers, then each frame contains a representation of the space time manifold as a $4-tuple$ of the real number representation.

To understand these observations better it is useful to briefly describe theories. Here the usual mathematical logic [9] characterization of theories as being defined by a set of axioms is used. All theories have in common the logical axioms and logical rules of deduction; they are distinguished by having different sets of nonlogical axioms. This is the case whether the axioms are explicitly stated or not.[1]

Each theory described by a consistent set of axioms has many representations (called models in mathematical logic)[2] as mathematical structures in which the theory axioms are true. Depending on the axiom set the representations may or may not be isomorphic.

The real numbers axiomatized as a complete ordered field are an example of a simple theory. For this axiomatization all representations of the real numbers are isomorphic. However they are not the same. All theories based on the real (or complex) numbers include the real (and complex) number axioms in

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[1] The importance of the axiomatic characterization of physical or mathematical theories is to be emphasized. All properties of physical or mathematical systems are described in the theory as theorems which are obtained from the axioms by use of the logical rules of deduction. Without axioms a theory is empty as it can not make any predictions or have any meaning.

[2] In physics models have a different meaning in that they are also theories. However they are simpler theories as they are based on simplifying model assumptions which serve as axioms for the simpler theory.

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their axiom sets. These well known aspects of theories are are quite familiar in the application of group theory to physics. Each abstract group is defined by a set of axioms that consist of the general group axioms and additional ones to describe the particular group considered. Each group has many different representations as different mathematical systems. These can be matrices or operators in quantum theory. These are further classified by the dimensionality of the representation and whether they are reducible or irreducible. The importance of different irreducible representations to describe physical systems and their states is well known.

As sets of axioms and derived theorems, physical theories, unlike mathematical theories, also have representations as descriptions of physical systems and their properties. It is immaterial here whether these representations are considered to be maps from theory statements to physical systems and their properties or from different mathematical representations of the theory to physical systems and their properties. It is, however, relevant to note that theoretical predictions to be tested by experiment are or can be represented by equations whose solutions are real numbers. Spectra of excited states of nuclei, atoms, and molecules are examples. The same holds for observables with discrete spectra such as spin, isospin, and angular momentum. Here the eigenvalues are the real number equivalents of integers or rational numbers.

3 Real Numbers and their Representation in Quantum Theory

It is useful to begin with a question "What are the real numbers?" The most general answer is that they are elements of any set of mathematical or physical systems that satisfies the real number axioms. The axioms express the requirements that the set must be a complete, ordered field. This means that the set must be closed under addition, multiplication and their inverses, A linear order must exist for the collection, and any convergent sequence of elements converges to an element of the set.

It follows that all sets of real numbers must at least have these properties. However they can have other properties as well. A study of most any mathematical analysis textbook will show that real numbers are defined as equivalence classes of either Dedekind cuts or of Cauchy sequences of rational numbers.[3] A sequence t_n of rational numbers is a Cauchy sequence if

$$
\begin{aligned}
&\text{For all } l \text{ there is an } h \text{ such that} \\
&|t_j - t_k| \leq 2^{-l} \\
&\text{for all } j, k > h.
\end{aligned}
\tag{1}
$$

The proof that the set of equivalence classes of Cauchy sequences are real numbers requires proving that it is a complete ordered field.

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[3] Rational numbers are defined as equivalence classes of ordered pairs of integers which are in turn defined as equivalence classes of ordered pairs of the natural numbers $0, 1, 2, \cdots$.

A similar situation holds for rational numbers (and integers and natural numbers). Rational numbers are elements of any set that satisfies the axioms for an ordered field. However the field is not complete.

The representation of rational numbers used here will be based on finite strings of digits or kits in some base k along with a sign and "$k-al$" point. Use of the string representation is based on the fact that physical representations of rational numbers (and integers and natural numbers) are given as digits or states of strings of kits or qukits (with a sign and "$k-al$" point for rational numbers) in some base $k \geq 2$. Such numbers are also the base of all computations, mainly because of the efficiency in representing large numbers and in carrying out arithmetic operations. The usefulness of this representation is based on the fact that for any base $k \geq 2$, these string numbers are dense in the set of all rational numbers.

Here, to keep things simple, representations will be limited to binary ones with $k = 2$. The representations will be further restricted here to be represented as states of finite strings of qubits. This is based on the fact that quantum mechanics is the basic mechanics applicable to all physical systems. The Cauchy condition will be applied to sequences of these states to give quantum theory representations of real numbers.

It should be noted that there are also quantum theory representations of real numbers that are not based on Cauchy sequences of states of qukit strings. Besides those described in [14, 15, 16] there are representations as Hermitian operators in Boolean valued models of ZF set theory [11, 12, 13] and in a category theory context [17]. These representations will not be considered further here because of the limitation here of representations to those based on finite strings of qubits.

3.1   Rational Number States

Here a compact representation of rational string states is used that combines the location of the "$binal$" point and the sign. For example, the state $|1001 - 0111\rangle$ is a state of eight $0 - 1$ qubits and one $\pm$ qubit representing a rational string number -9.4375 in the ordinary decimal form.

Qubit strings and their states can be described by locating qubits on an integer lattice[4] Rational string states correspond to states of qubits occupying an integer interval $[m + l, m + u]$. Here $l \leq 0 \leq u$, m is the location of the $\pm$ qubit, and the $0-1$ qubits occupy all positions in the interval $[m+l, m+u]$. Note that two qubits, a sign one and a $0 - 1$ one, occupy position m. For fermionic qubits this can be accounted for by including extra variables to distinguish the qubit types.

One way to describe rational string states is by strings of qubit annihilation creation (AC) operators $a_{\alpha,j}, a_{\alpha,j}^{\dagger}$ acting on a qubit vacuum state $|0\rangle$. Also

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[4]Note that the only relevant properties of the integer locations is their ordering, a discrete linear ordering. Nothing is assumed about the spacing between adjacent locations on the lattice.

present is another qubit type represented by the AC operators $c_{\gamma,m} c_{g,m}^\dagger$. Here $\alpha = 0, 1; \gamma = +, -$, and j, m are integers.

For this work it is immaterial whether the AC operators satisfy commutation relations or anticommutation relations:

$$[a_{\alpha,j}, a_{\alpha',j'}^\dagger] = \delta_{j,j'} \delta_{\alpha,\alpha'}$$
$$[a_{\alpha,j}^\dagger, a_{\alpha',j'}^\dagger] = [a_{\alpha,j}, a_{\alpha',j'}] = 0 \tag{2}$$

or

$$\{a_{\alpha,j}, a_{\alpha',j'}^\dagger\} = \delta_{j,j'} \delta_{\alpha,\alpha'}$$
$$\{a_{\alpha,j}^\dagger, a_{\alpha',j'}^\dagger\} = \{a_{\alpha,j}, a_{\alpha',j'}\} = 0. \tag{3}$$

with similar relations for the c operators. The c operators commute with the a operators.

Rational number states are represented by strings of a creation operators and one c creation operator acting on the qubit vacuum state $|0\rangle$ as

$$|\gamma, m, s, l, u\rangle = c_{\gamma,m}^\dagger a_{s(u),u}^\dagger \cdots a_{s(l),l}^\dagger |0\rangle. \tag{4}$$

Here $l \leq m \leq u$ and $l < u$ with l, m, u integers, and $s : [l, u] \to \{0, 1\}$ is a $\{0, 1\}$ valued function on the integer interval $[l, u]$. Alternatively s can be considered as a restriction to $[l, u]$ of a function defined on all the integers.

An operator $\tilde{N}$ can be defined whose eigenvalues correspond to the values of the rational numbers one associates with the string states. $\tilde{N}$ is the product of two commuting operators, a sign scale operator $\tilde{N}_{ss}$, and a value operator $\tilde{N}_v$. One has

$$\tilde{N} = \tilde{N}_{ss} \tilde{N}_v$$
$$\text{where } \tilde{N}_{ss} = \sum_{\gamma,m} \gamma 2^{-m} c_{\gamma,m}^\dagger c_{\gamma,m} \tag{5}$$
$$\tilde{N}_v = \sum_{i,j} i 2^j a_{i,j}^\dagger \alpha_{i,j}.$$

The operator is given for reference only as it is not used to define arithmetic properties of the rational string states.

There is a large amount of arithmetical redundancy in the states. For instance the arithmetic properties of a rational string state are invariant under a translation along the integer axis. This is a consequence of the fact that these properties of the state are determined by the distribution of $1s$ relative to the position m of the sign and not on the value of m. The other redundancy arises from the fact that states that differ in the number of leading or trailing $0s$ are all arithmetically equal.

These redundancies can be used to define equivalence classes of states or select one member as a representative of each class. Here the latter choice will be made in that rational number states will be restricted to those with $m = 0$ for the sign location and those with s restricted so that $s(l) = 1$ if $l < 0$ and $s(u) = 1$ if $u > 0$. This last condition removes leading and trailing $0s$. The state $a_{0,0}^\dagger c_{+,0}^\dagger |0\rangle$ is the number 0. For ease in notation from now on the variables m, l, u will be dropped from states. Thus states $|0, \gamma, s, l, u\rangle$ will be represented as $|\gamma, s\rangle$ with the values of l, u included in the definition of s.

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There are two basic arithmetic properties, equality, $=_A$ and ordering $\leq_A$ Arithmetic equality is defined by

$$|\gamma, s\rangle =_A |\gamma', s'\rangle,$$
$$\text{if } l' = l, \; u' = u, \; \gamma' = \gamma \;\text{ and } 1_{s'} = 1_s. \tag{6}$$

Here $1_s = \{j : s(j) = 1\}$ the set of integers j for which $s(j) = 1$ and similarly for $1_{s'}$. That is, two states are arithmetically equal if one has the same distribution of $1s$ relative to the location of the sign as the other.

Arithmetic ordering on positive rational string states is defined by

$$|+, s\rangle \leq_A |+, s'\rangle, \;\text{ if } 1_s \leq 1_{s'} \tag{7}$$

where
$$1_s < 1_{s'} \text{ if there is a } j \text{ where } j \text{ is in } 1_{s'} \text{ and not in } 1_s$$
$$\text{and for all } m > j, m\epsilon 1_s \text{ iff } m\epsilon 1_{s'}. \tag{8}$$

The extension to zero and negative states is given by

$$|+, \underline{0}\rangle \leq_A |+, s\rangle \text{ for all } s$$
$$|+, s\rangle \leq_A |+, s'\rangle \rightarrow |-, s'\rangle \leq_A |-, s\rangle. \tag{9}$$

The definitions of $=_A, \leq_A$ can be extended to linear superpositions of rational string states in a straightforward manner to give probabilities that two states are arithmetically equal or that one state is less than another state.

Operators for the basic arithmetic operations of addition, subtraction, multiplication, and division to any accuracy $|+, -\ell\rangle$ are represented by $\tilde{+}_A, \tilde{-}_A, \tilde{\times}_A, \tilde{\div}_{A,\ell}$. The state

$$|+, -\ell\rangle = c^\dagger_{+,0} a^\dagger_{\underline{0}_{[0,-\ell+1]}} a^\dagger_{1,-\ell} |0\rangle. \tag{10}$$

where $a^\dagger_{\underline{0}_{[0,-\ell+1]}} = a^\dagger_{0,0} a^\dagger_{0,-1} \cdots a^\dagger_{0,-\ell+1}$, is an eigenstate of $\tilde{N}$ with eigenvalue $2^{-\ell}$.

As an example of the explicit action of the arithmetic operators, the unitary addition operator $\tilde{+}$ satisfies

$$\tilde{+}_A |\gamma s\rangle |\gamma' s'\rangle = |\gamma s\rangle |\gamma'' s''\rangle \tag{11}$$

where $|\gamma'' s''\rangle$ is the resulting addend state. It is often useful to write the addend state as

$$|\gamma'' s''\rangle = |\gamma' s' +_A \gamma s\rangle =_A |\gamma', s'\rangle +_A |\gamma, s\rangle. \tag{12}$$

More details on the arithmetic operations are given elsewhere [3]. Note that these operations are quite different from the usual quantum theory superposition, product, etc. operations. This is the reason for the presence of the subscript A.

3.2 The Cauchy Condition

The arithmetic operators can be used to define rational number properties of rational string states and their superpositions. They can also be used to define the Cauchy condition for a sequence of rational string states. Let $\{|\gamma_n, s_n\rangle : n = 1, 2, \cdots\}$ be any sequence of rational string states. Here for each n $\gamma_n \epsilon \{+, -\}$ and s_n is a $0-1$ valued function from a finite integer interval that includes 0.

The sequence $\{|\gamma_n s_n\rangle\}$ satisfies the Cauchy condition if

$$\text{For each } \ell \text{ there is an } h \text{ where for all } j, k > h \qquad (13)$$
$$|(|\gamma_j s_j -_A \gamma_k s_k|_A)\rangle <_A |+, -\ell\rangle.$$

In this definition $|(|\gamma_j s_j -_A \gamma_k s_k|_A)\rangle$ is the state that is the arithmetic absolute value of the arithmetic difference between the states $|\gamma_j, s_j\rangle$ and $|\gamma_k, s_k\rangle$. The Cauchy condition says that this state is arithmetically less than or equal to the state $|+, -\ell\rangle$ for all j, k greater than some h.

It must be emphasized that this Cauchy condition statement is a direct translation of Eq. 1 to apply to rational string states. It has nothing to do with the usual convergence of sequences of states in a Hilbert or Fock space. It is easy to see that state sequences which converge arithmetically do not converge quantum mechanically.

It was also seen in [3] that the Cauchy condition can be extended to sequences of linear superpositions of rational states. Let $\psi_n = \sum_{\gamma, s} |\gamma, s\rangle\langle\gamma s|\psi_n\rangle$. Here $\sum_s = \sum_{l \leq 0} \sum_{u \geq 0} \sum_{s_{[l,u]}}$ is a sum over all integer intervals $[l, u]$ and over all $0-1$ valued functions from $[l, u]$. From this one can define the probability that the arithmetic absolute value of the arithmetic difference between ψ_j and ψ_k is arithmetically less than or equal to $|+, -\ell\rangle$ by

$$P_{j,m,\ell} = \sum_{\gamma, s} \sum_{\gamma', s'} |\langle\gamma, s|\psi_j\rangle\langle\gamma', s'|\psi_m\rangle|^2 \qquad (14)$$
$$|(|\gamma, s -_A \gamma', s'|_A)\rangle \leq_A |+, -\ell\rangle.$$

The sequence $\{\psi_n\}$ satisfies the Cauchy condition if $P_{\{\psi_n\}} = 1$ where

$$P_{\{\psi_n\}} = \liminf_{\ell \to \infty} \limsup_{h \to \infty} \liminf_{j,k>h} P_{j,m,\ell}. \qquad (15)$$

Here $P_{\{\psi_n\}}$ is the probability that the sequence $\{\psi_n\}$ satisfies the Cauchy condition.

Cauchy sequences can be collected into equivalence classes by defining $\{|\gamma_n, s_n\rangle\} \equiv \{|\gamma'_n s'_n\rangle\}$ if the Cauchy condition holds with γ'_k replacing γ_k and s'_k replacing s_k in Eq. 13. To this end let $[\{|\gamma_n, s_n\rangle\}]$ denote the equivalence class containing the Cauchy sequence $\{|\gamma_n, s_n\rangle\}$. Similarly $\{\psi_n\} \equiv \{\psi'_m\}$ if $P_{\{\psi_n\} \equiv \{\psi'_m\}} = 1$ where $P_{\{\psi_n\} \equiv \{\psi'_m\}}$ is given by Eqs. 14 and 15 with ψ'_k replacing ψ_k in Eq. 14.

The definitions of $=_A, \leq_A, \tilde{+}_A, \tilde{-}_A, \tilde{\times}_A, \tilde{\div}_{A,\ell}$ can be lifted to definitions of $=_R, \leq_R, \tilde{+}_R, \tilde{-}_R, \tilde{\times}_R, \tilde{\div}_R$ on the set $[\{|\gamma_n, s_n\rangle\}]$ of all equivalence classes. It can be shown [?] that $[\{|\gamma_n, s_n\rangle\}]$ with these operations and relations is a representation or model of the real number axioms. In this sense it is entirely equivalent to R which is the real number component of the complex number base for the

Hilbert space containing the rational string number states that were used to define $[\{|\gamma_n, s_n\rangle\}]$.

Another representation of real numbers can be obtained by replacing the sequences $\{|\gamma_n, s_n\rangle\}$ by operators. This can be achieved by replacing each index n by the rational string state that corresponds to the natural number n. These are defined by $|\gamma, s\rangle$ where $\gamma = +$ and $l = 0$ where l is the lower interval bound, for the domain of s as a $0 - 1$ function over the integer interval $[l, u]$.

In this case each sequence $|\gamma_n, s_n\rangle$ corresponds to an operator $\tilde{O}$ defined on the domain of natural number states. One has

$$\tilde{O}|+, s\rangle = |\gamma_n, s_n\rangle \tag{16}$$

where n is the $\tilde{N}$ (defined in Eq. 5) eigenvalue of the state $|+, s\rangle$. $\tilde{O}$ is defined to be Cauchy if the righthand sequence in Eq. 16 is Cauchy. One can also give a Cauchy condition for $\tilde{O}$ by replacing the natural numbers in the definition quantifiers by natural number states.

One can repeat the description of equivalence classes of Cauchy sequences of states for the operators to obtain another representation of real numbers as equivalence classes of Cauchy operators. The two definitions are closely related as Eq. 16 shows, and should be equivalent as representations of real number axioms. This should follow from the use of Eq. 16 to replace the left hand expression for the right hand expression in all steps of the proofs that Cauchy sequences of rational string states satisfy the real number axioms.

4 Gauge Transformations

The representation of rational string states as states of strings of qubits as in Eq. 4 implies a choice of quantization axes for each qubit. Usually one assumes the same axis for each qubit where the axis is fixed by some external field. However this is not necessary, and in some cases, such as quantum cryptography [?], rotation of the axis plays an important role.

In general there is no reason why the axes cannot be arbitrarily chosen for each qubit. This freedom of arbitrary directions for the axes of each qubit corresponds to the set of possible local and global gauge transformations of the qubit states. Each gauge transformation corresponds to a particular choice in that it defines the axis direction of a qubit relative to that of its neighbors.

Here a gauge transformation U can be defined as an $SU(2)$ valued function on the integers, $U : \{\cdots - 1, 0, 1, \cdots\} \rightarrow SU(2)$. U is global if U_j is independent of j, local if it depends on j. The effect of U on a rational string state $|\gamma, s\rangle$ is given by

$$U|\gamma, s\rangle = U_0 c^\dagger_{\gamma,0} U_u a^\dagger_{s(u),u} \cdots U_l a^\dagger_{s(l),l}|0\rangle = (c^\dagger_{U_0})_{\gamma,0}(a^\dagger_{U_u})_{s(u),u} \cdots (a^\dagger_{U_l})_{s(l),l}|0\rangle \tag{17}$$

where

$$\begin{aligned}(a^\dagger_{U_j})_{i,j} &= U_j a^\dagger_{i,j} = \sum_k (U_j)_{i,k} a^\dagger_{k,j} \\ a_{U_j})_{h,j} &= a_{h,j} U^\dagger_j = \sum_i (U_j)^*_{i,h} a_{i,j}\end{aligned} \tag{18}$$

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These results are based on the representation of U_j as

$$U_j = \sum_{i,h} (U_j)_{i,h} a_{i,j}^\dagger a_{h,j} \tag{19}$$

Arithmetic relations and operators transform in the expected way. For the relations one defines $=_{A,U}$ and $\leq_{A,U}$ by

$$\begin{aligned} =_{A,U} &:= (U =_A U^\dagger) \\ \leq_{A,U} &:= U \leq_A U^\dagger. \end{aligned} \tag{20}$$

These relations express the fact that $U|\gamma s\rangle =_{AU} U|\gamma' s'\rangle$ if and only if $|\gamma, s\rangle =_A |\gamma', s'\rangle$ and $U|\gamma, s\rangle \leq_{AU} U|\gamma', s'\rangle$ if and only if $|\gamma, s\rangle \leq_A |\gamma', s'\rangle$.

For the operation $\tilde{+}_A$ one defines $\tilde{+}_{A,U}$ by

$$\tilde{+}_{A,U} := (U \times U)\tilde{+}_A(U^\dagger \times U^\dagger). \tag{21}$$

Then

$$\begin{aligned} &\tilde{+}_{A,U}(U|\gamma, s\rangle \times U|\gamma', s'\rangle) \\ &\quad = (U \times U)\tilde{+}_A(|\gamma, s\rangle \times |\gamma', s'\rangle) \end{aligned} \tag{22}$$

as expected. This is consistent with the definition of $\tilde{+}_A$ in Eq. 11 as a binary relation. Similar relations hold for $\tilde{\times}_A, \tilde{-}_A, \tilde{\div}_{A,l}$.

It follows from these properties that the Cauchy condition is preserved under gauge transformations. If a sequence of states $\{\psi_n\}$ is Cauchy then the sequence $\{U\psi_n\}$ is U-Cauchy which means that it is Cauchy relative to the transformed arithmetic relations and operations. For example, if a sequence $\{|\gamma_n, s_n\rangle\}$ satisfies the Cauchy condition, then the transformed sequence $\{U|\gamma_n, s_n\rangle\}$ satisfies the U-Cauchy condition:

$$\begin{aligned} &\text{For each } \ell \text{ there is an } h \text{ where for all } j, k > h \\ &|(U|\gamma_j s_j -_{AU} U\gamma_k s_k|_{AU})\rangle <_{AU} U|+, -\ell\rangle. \end{aligned} \tag{23}$$

These definitions and considerations extend to the Cauchy operators. If $\tilde{O}$ is Cauchy the the above shows that

$$\tilde{O}_U = U\tilde{O}U^\dagger \tag{24}$$

is U-Cauchy. However $\tilde{O}_U$ is not a Cauchy operator in the original frame and $\tilde{O}$ is not Cauchy in the transformed frame.

To see that $\tilde{O}_U$ is not Cauchy in the original frame it is instructive to consider a simple example. First one works with the Cauchy property for sequences of states. Let $f : (-\infty, n] \to \{0, 1\}$ be a $0-1$ function from the set of all integers $\leq n$ where $f(n) = 1$. Define a sequence of states

$$|f\rangle_m = c_{+,0}^\dagger a_{f(n),n}^\dagger a_{f(n-1),n-1}^\dagger \cdots a_{f(-m),-m}^\dagger |0\rangle \tag{25}$$

for $m = 1, 2, \cdots$. The sequence is Cauchy as $||f(j)\rangle -_A |f(k)\rangle|_A \leq_A |+, -\ell\rangle$ for all $j, k > \ell$. However for any gauge transformation U the sequence

$$U|f\rangle_m = c_{+,0}^\dagger (a_U^\dagger)_{f(n),n} \cdots (a_U^\dagger)_{f(-m),-m}|0\rangle \tag{26}$$

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is not Cauchy as expansion of the $a_U^\dagger$ in terms of the $a^\dagger$ by Eq. 18 gives $U|f\rangle_m$ as a sum of terms whose arithmetic divergence is independent of m.

To show that $\tilde{O}_U$ is Cauchy in the transformed frame if and only if $\tilde{O}$ is Cauchy in the original frame one can start with the expression for the Cauchy condition in the transformed frame:

$$|\tilde{O}_U U|s_j\rangle -_{AU} \tilde{O}_U U|s_k\rangle|_{AU} \leq_{AU} U|+, -\ell\rangle$$
$$\text{for all } U|s_j\rangle, \, U|s_k\rangle \geq_{AU} \text{ some } U|s_h\rangle \tag{27}$$

From Eq. 24 one gets

$$\tilde{O}_U U|s_j\rangle -_{AU} \tilde{O}_U U|s_k\rangle = U\tilde{O}|s_j\rangle -_{AU} U\tilde{O}|s_k\rangle.$$

From Eqs. 11 and 12 applied to $-_{AU}$ and Eqs. 21 and 22 one obtains

$$U\tilde{O}|s_j\rangle -_{AU} U\tilde{O}|s_k\rangle = U(\tilde{O}|s_j\rangle -_A \tilde{O}|s_k\rangle).$$

Use of

$$|-|_{AU} = U|-|_A U^\dagger \tag{28}$$

for the absolute value operator gives

$$|U(\tilde{O}|s_j\rangle -_A \tilde{O}|s_k\rangle)|_{AU} = U(|\tilde{O}|s_j\rangle -_A \tilde{O}|s_k\rangle|_A).$$

Finally from Eq. 20 one obtains

$$U(|\tilde{O}|s_j\rangle -_A \tilde{O}|s_k\rangle|_A) \leq_{AU} U|+, -\ell\rangle$$
$$\leftrightarrow |\tilde{O}|j\rangle -_A \tilde{O}|s_k\rangle|_A \leq_A |+, -\ell\rangle$$

which is the desired result. Thus one sees that the Cauchy property is preserved in unitary transformations from one reference frame to another.

As was done with the Cauchy sequences and operators, the U-Cauchy sequences or their equivalents, U-Cauchy operators, can be collected into a set $\mathcal{R}_U$ of equivalence classes that represent the real numbers. This involves lifting up the basic arithmetic relations $=_{AU}, \leq_{AU}$ and operations $\tilde{+}_{AU}, \tilde{\times}_{AU}, \tilde{-}_{AU}, \tilde{\div}_{AU,l}$ to real number relations $=_{RU}, \leq_{RU}$ and operations $\tilde{+}_{RU}, \tilde{\times}_{RU}, \tilde{-}_{RU}, \tilde{\div}_{RU}$, and showing that $\mathcal{R}_U$ is a complete, ordered, field.

It is also the case that for almost all gauge U the real numbers in $\mathcal{R}_U$ are orthogonal to those in $\mathcal{R}$ in the following sense. One can see that each equivalence class in $\mathcal{R}$ contains a state sequence $|\gamma_n, s_{[u,-n]}\rangle$ where s is a $0 - 1$ valued function on the interval of all integers $\leq u$. Let U be a gauge transformation with associated state sequence $|U_0\gamma_n, Us_{[u,-n]}\rangle$. Both sequences satisfy their respective Cauchy conditions. However the overlap $\langle \gamma_n, s_{[u,-n]}|U_0\gamma_n, Us_{[u,-n]}\rangle \to 0$ as $n \to \infty$. This expresses the sense in which $\mathcal{R}$ and $\mathcal{R}_U$ are orthogonal.

5   Fields of Quantum Frames

As has been seen, one can define many quantum theory representations of real numbers as Cauchy sequences of states of qubit strings or as Cauchy operators

on the qubit string states. The large number of representations stems from the gauge (global and local) freedom in the choice of a quantization axis for the qubit strings. Complex number representations are included either as ordered pairs of the real number representations or as extensions of the description of Cauchy sequences or Cauchy operators to complex rational string states [3].

It was also seen that for each gauge transformation U the real and complex number representations $\mathcal{R}_U$, $\mathcal{C}_U$ are the base of a frame F_U. The frame F_U also contains representations of physical theories as mathematical structures based on $\mathcal{R}_U$, $\mathcal{C}_U$.

The work in the last two sections shows that the description of rational string states as states of finite strings of qubits given is a description in a Fock space. (A Fock space is used because of the need to describe states of variable numbers of qubits and their linear superpositions in one space.) The arithmetic operations $+_A, -_A, \times_A, \div_{A,\ell}$ on states of these strings are represented by Fock space operators. The properties of these operators acting on the qubit string states are used to show that the states represent binary rational numbers. Finally equivalence classes of sequences of these states or of operators that satisfy the Cauchy condition are proved to be real numbers [3].

The essential point here is that the Fock space, $\mathcal{F}$, and any additional mathematics used to obtain these results are based on a set R, C of real and complex numbers. For example, superposition coefficients of basis states are in C, the inner product is a map from pairs of states to C, operator spectra are elements of C, etc. The space time manifold used to describe the dynamics of any physical representations of the qubit strings is given by R^4.

It follows that one can assign a reference frame F to R and C. Here F contains all physical and mathematical theories that are represented as mathematical structures based on R and C. However, unlike the case for the frames F_U, the only properties that R and C have are those based on the relevant axioms (complete ordered field for R). Other than that, nothing is known about how they are represented.

This can be expressed by saying that R and C are external, absolute, and given. This seems to be the usual position taken by physics in using theories based on R and C. Physical theories are silent on what properties R and C may have other than those based on the relevant axioms. However, as has been seen, one can use these theories to describe many representations R_U and C_U and associated frames F_U based on $SU(2)$ gauge transformations of the qubit strings. As noted, for each U, F_U contains representations of all physical theories as mathematical structures over R_U, C_U. For these frames one can see that R_U and C_U have additional properties besides those given by the relevant axioms. They are also equivalence classes of Cauchy sequences $\{U|\gamma_n, s_n\rangle\}$ or Cauchy operators $\tilde{O}_U$.

Fig. 1 is a schematic illustration of the relation between frame F and the frames F_U. Only three of the infinitely many F_U frames are shown. The arrows indicate the derivation direction in that R, C based theory in F is used to describe, for each U R_U and C_U that are the base of all theories in F_U. Note that the frame F_{ID} with the identity gauge transformation is also included as

one of the F_U. It is *not* the same as F because R_{ID} is not the same as R.
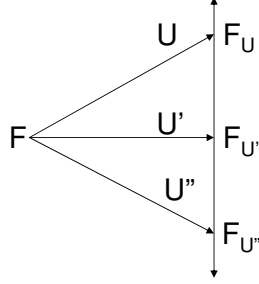

Figure 1: Relation between a Base Frame F and Gauge Transformation Frames. A frame F_U is associated with each gauge transformation U of the rational string states in F. The three frame connections shown are illustrative of the infinitely many connections, shown by the two headed vertical arrow. Each F_U is based on real and complex numbers $\mathcal{R}_U, \mathcal{C}_U$ and a space time manifold $\mathcal{R}_U^4$.

The above relations between the frames F and F_U shows that one can repeat the description of real and complex numbers as Cauchy sequences of (or Cauchy operators on) rational string states in each frame F_U. In this case the Fock space representation, $\mathcal{F}_U$, used to describe the qubit string states in F_U, is different from from $\mathcal{F}$ in that it is based on R_U, C_U instead of on R, C. However the two space representations are related by an isomorphism that is based on the isomorphism between R and R_U.

It is useful to examine this more. First consider the states of a qubit at site j. For an observer in frame F, these states have the general form $\alpha|0\rangle + \beta|1\rangle$ where α and β are complex numbers in C. Let U be an $SU(2)$ gauge transformation where $U(j)$ is defined by

$$U(j)|0\rangle = |+\rangle$$
$$U(j)|1\rangle = |-\rangle$$

where $|\pm\rangle = (1/\sqrt{2})|0\rangle \pm |1\rangle)$. Then the states $|+\rangle$ and $|-\rangle$ in frame F are seen by an observer in F_U as the states $|0\rangle$ and $|1\rangle$ respectively as the quantization axis is different.

A similar situation holds for states of qubit strings. To an observer in F the state $U|\gamma, s\rangle$ is different from $|\gamma, s\rangle$. However an observer in frame F_U, with a different set of quantization axes for each qubit, would represent the state $U|\gamma, s\rangle$ as $|\gamma, s\rangle$ as to him it is the same state relative to his axis set as it is to the observer in F for his axis set.

The situation is slightly different for linear superpositions of basis states. To an observer in F, the coefficients α, β in the state $\alpha U|\gamma, s\rangle + \beta U|\gamma', s'\rangle$ represent abstract elements of C. The same observer sees that this state in F_U is represented by $\alpha_U|\gamma, s\rangle + \beta_U|\gamma', s'\rangle$ where α_U, β_U as elements of C_U, represent

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the same abstract complex numbers as do α, β. However an observer in F_U sees this same state as $\alpha|\gamma, s\rangle + \beta|\gamma', s'\rangle$. To him the real and complex number base of his frame is abstract and is represented by R, C.

In general an observer in any frame sees the real and complex number base of his own frame as abstract and given with no particular representation evident. However the observer in frame F also sees that what to him is the abstract number r, is the number r_U as and element of R_U in frame F_U.

These considerations also extend to group representations as matrices of complex numbers. If the element g, as an abstract element of $SU(2)$, is represented in frame F by the matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where a, b, c, d are elements of C, then, to an observer in F, g is represented in frame F_U by $\begin{vmatrix} a_U & b_U \\ c_U & d_U \end{vmatrix}$. Here a_U, b_U, c_U, d_U, as elements of C_U, correspond to the same abstract complex numbers as do a, b, c, d. However an observer in F_U sees this representation as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ which is the same as the F observer sees in his own frame.

Following this line of argument one can now describe another generation of frames with each frame F_U in the role of a parent to progeny frames just as F is a parent to the frames F_U as in Fig. 1. This is shown schematically in Fig. 2. Again, only three of the infinitely many stage 2 frames emanating from each stage 1 frame are shown. Here something new appears in that there are many
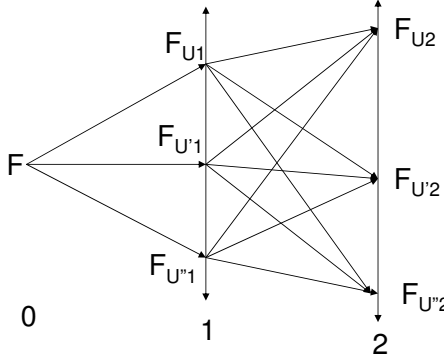

Figure 2: Three Iteration Stages of Frames coming from Frames. Only three frames of the infinitely many, one for each gauge U, are shown at stages 1 and 2. The arrows connecting the frames show the iteration direction of frames emanating from frames.

different paths to a stage 2 frame. For each path, such as $F \to F_{U_1} \to F_{U_2}$, U_2 is the product $U''U_1$ of two gauge transformations where $U'' = U_2 U_1^\dagger$. An observer in this frame F_{U_2} sees the real and complex number frame base as abstract, and given. To him they can be represented as R, C. An observer in F_{U_1} sees the real and complex number base of F_{U_2} as $R_{U''}, C_{U''}$.

However an observer in F sees the real and complex number base of F_{U_2} as $R_{U''|U_1}, C_{U''|U_1}$. The subscript $U''|U_1$ denotes the fact that relative to F the number base of F_{U_2} is constructed in two stages. First the Fock space $\mathcal{F}$ is used to construct representations R_{U_1}, C_{U_1} of R and C as U_1 Cauchy sequences of states $\{U_1|\gamma_n, s_s\rangle\}$. Then in frame F_{U_1} the Fock space $\mathcal{F}_{U_1}$, based on R_{U_1}, C_{U_1}, is used to construct the number representation base of F_{U_2} as U'' Cauchy sequences $\{U''|\gamma_n, s_n\rangle\}$ of qubit string states in $\mathcal{F}_{U_1}$.

One sees, then, that, for each path leading to a specific stage 2 frame, there is a different two stage construction of the number base of the frame. This view from the parent frame F is the same view that we have as observers outside the whole frame structure. That is, our external view coincides with that for an observer inside the parent frame F.

The above description of frames emanating from frames for 2 stages suggests that the process can be iterated. There are several possibilities besides a finite number of iterations exemplified by Fig.2 for 2 iterations. Fig. 3 shows the field structure for a one way infinite number of iterations. Here one sees that
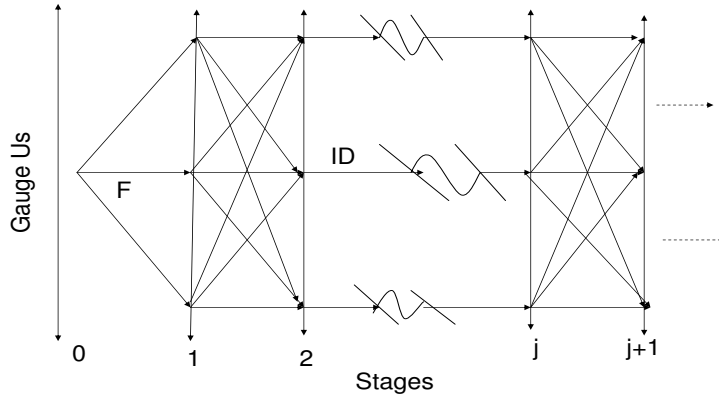

Figure 3: One way Infinite Iteration of Frames coming from Frames. Only three of the infinitely many frames, one for each gauge U are shown for stages $1, 2, \cdots, j, j+1, \cdots$. The arrows connecting the frames show the iteration or emanation direction. The center arrows labeled ID denote iteration of the identity gauge transformation.

each frame has an infinite number of descendent frame generations and, except for frame F, a finite number number of ancestor generations. The structure of the frame field seen by an observer in F is the same as that viewed from the outside. For both observers the base real and complex numbers for F are seen as abstract and given with no structure other than that given by the axioms for the real and complex numbers.

There are two other possible stage or generation structures for the frame fields, two way infinite and finite cyclic structures. These are shown schematically in Figs. 4 and 5. The direction of iteration in the cyclic field is shown by the arrows on the circle rather than example arrows connecting frames as in the

other figures. For both these frame fields each frame has infinitely many parent frames and infinitely many daughter frames. There is no single ancestor frame and no final descendent frames. The cyclic field is unique in that each frame is both its own descendent and its own ancestor. The distance between these connections is equal to the number of iterations in the cyclic field.
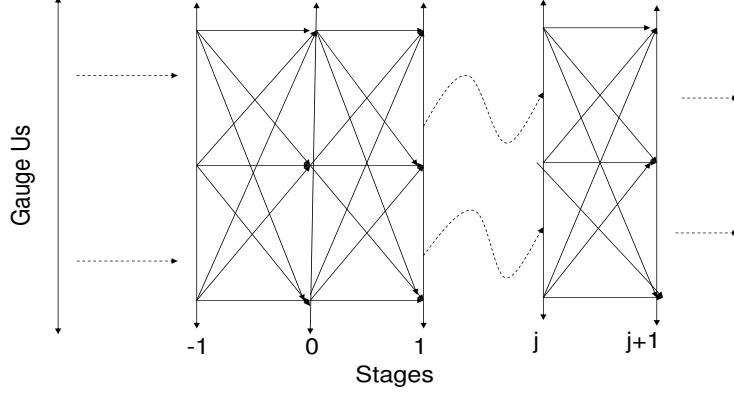

Figure 4: Two way Infinite Iteration of Frames coming from Frames. Only three of the infinitely many frames, one for each gauge U are shown at each stage $\cdots, -1, 0, 1, \cdots, j, j + 1, \cdots$. The solid arrows connecting the frames show the iteration or emanation direction. The wavy arrows denote iterations connecting stage 1 frames to those at stage j. The straight dashed arrows denote infinite iterations from the left and to the right.of the identity gauge transformation.
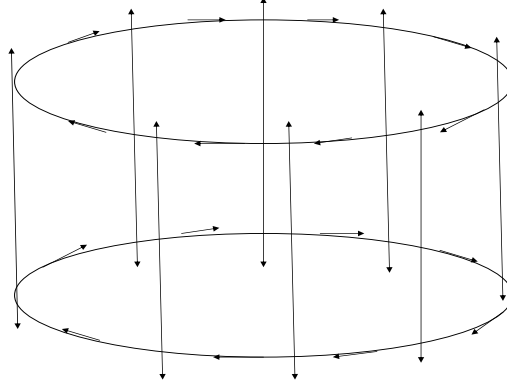

Figure 5: Schematic Representation of Cyclic Iteration of Frames coming from Frames. The vertical two headed arrows represent the gauge transformations at each stage and the arrows along both ellipses show the direction of iteration. To avoid a very complex and messy figure no arrows connecting frames to frames are shown.

These two frame fields differ from the others in that the structure seen from

the outside is different from that for an observer in any frame. There is no frame F from which the view is the same as from the outside. Viewed from the outside there are no abstract, given real and complex number sets for the field as a whole. All of these are internal in that an observer in frame F_{U_j} at generation j sees the base R_{U_j}, C_{U_j} as abstract with no properties other than those given by the relevant axioms.

The same holds for the representations of the space time manifold. Viewed from the outside there is no fixed abstract space time representation as a 4-tuple of real numbers associated with the field as a whole. All space time representations of this form are internal and associated with each frame. This is based on the observation that the points of a continuum space time as a 4-tuple of representations of the real numbers are different in each frame because the real number representations are different in each frame. Also, contrary to the situation for the fields in Figs. 1-3, there is no representation of the space time points that can be considered to be as fixed and external to the frame field.

The lack of a fixed abstract, external space time manifold representation for the two-way infinite and cyclic frame fields is in some ways similar to the lack of a background space time in loop quantum gravity [20]. There are differences in that in loop quantum gravity space is discrete on the Planck scale and not continuous [21]. It should be noted though that the representation of space time as continuous is not a necessary consequence of the frame fields and their properties. The important part is the real and complex number representation base of each frame, not whether space time is represented as discrete or continuous.

It is useful to summarize the views of observers inside frames and outside of frames for the different field types. For all the fields except the cyclic ones an observer in any frame F_{U_j} at stage j sees the real number base R_{U_j} of his frame as abstract and external with no properties other than those given by the axioms for a complete ordered field. The observer also cannot see any ancestor frames. He/she can see the whole frame field structure for all descendent frames at stages $k > j$. Except as noted below, the view of an outside observer is different in that he/she can see the whole frame field structure. This includes the point that, to internal observers in a frame, the real and complex number base of the frame is abstract and external.

For frame fields with a fixed ancestor frame F, Figs. 1, 2, 3, the view of an outside observer is almost the same as that of an observer in frame F. Both see the real and complex number base of F as abstract and external. Both can also see the field structure for all frames in the fields. However the outside observer can see that frame F has no ancestors. This is not available to an observer in F as he/she cannot see the whole frame field.

The cyclic frame field is different in that for an observer in any frame at stage j, frames at other stages are both descendent and ancestor frames. This suggests that the requirement that a frame observer cannot see the field structure for ancestor frames, but can see it for descendent frames, may have to be changed, at least for this type of frame field. How and what one learns from such changes are a subject for future work.

6   Relation between the Frame Field and Physics

So far, frame fields based on different quantum mechanical representations of real and complex numbers have been described. Each frame contains a representation of physical theories as mathematical structures based on the real number representation base of the frame. The representations of the physical theories in the different frames are different because the real (and complex) number representations are different. They are also isomorphic because the real (and complex) number representations are isomorphic.

The description of the frame field given so far is incomplete because nothing has been said about the relation of the frame field to physics. So far the only use of physics has been to limit the description of rational number representations to quantum mechanical states of strings of qubits.

The main problem here is that to date all physical theories make no use of this frame field. This is evidenced by the observation that the only properties of real numbers relevant to physical theories are those derivable from the real number axioms for a complete, ordered field. So far physical theories are completely insensitive to details of different representations of the real numbers.

This problem is also shown by the observation that there is no evidence of this frame structure and the multiplicity of real number representations in our view of the existence of just one physical universe with its space time manifold, and with real and complex numbers that can be treated as absolute and given. There is no hint, so far, either in physical theories or in properties of physical systems and the physical universe, of the different representations and the frame fields.

This shows that the main problem is to reconcile the great number of different representations of the real and complex numbers and the R^4 space time manifold as bases for different representations of physical theories with the lack of dependence of physical theories on these representations and no evidence for them in our view of the physical universe.

One possible way to do this might be to collapse the frame field to a smaller field, ideally with just one frame. As a step in this direction one could limit quantum theory representations of rational string numbers to those that are gauge invariant. This would have the effect of collapsing all frames F_{U_j} at any stage j into one stage j frame[5]. The resulting frame field would then be one dimensional with one frame at each stage.

The idea of constructing representations that are gauge invariant for some gauge transformations has already been used in another context. This is the use of the decoherent free subspace (DFS) approach to quantum error correction. This approach [18, 19] is based to quantum error avoidance in quantum computations. This method identifies quantum errors with gauge transformations U. In this case the goal is to find subspaces of qubit string states that are invariant under at least some gauge U and are preserved by the Hamiltonian dynamics for the system.

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[5]Note that U_j is a gauge transformation. It is not the *jth* element of one.

One way to achieve this is based on irreducible representations of direct products of $SU(2)$ as the irreducible subspaces are invariant under the action of some U. As an example, one can show that [8] the subspaces defined by the irreducible 4 dimensional representation of $SU(2) \times SU(2)$ are invariant under the action of any global U. The subspaces are the three dimensional subspace with isospin $I = 1$, spanned by the states $|00\rangle, |11\rangle, 1/\sqrt{2}(|01\rangle + |10\rangle)$ and the $I = 0$ subspace containing $1/\sqrt{2}(|01\rangle - |10\rangle)$. The action of any global U on states in the $I = 1$ subspace can change one of the I_z states into linear superpositions of all states in the subspace. But it does not connect the states in the $I = 1$ subspace with that in the $I = 0$ subspace.

It follows that one can replace a string of $2n$ qubits with a string of n new qubits where the $|0\rangle$ and $|1\rangle$ states of the jth new qubit correspond to any state in the respective $I = 1$ and $I = 0$ subspaces of the 4 dimensional representation of $SU(2)_{2j-1} \times SU(2)_{2j}$. Any state of the n new qubits is invariant under all global gauge transformations and all local gauge transformations where

$$U_{2j-1} = U_{2j}. \tag{29}$$

This replacement of states of strings of $2n$ qubits by states of strings of n new qubits gives the result that, for all U satisfying Eq. 29, the F_U frames at any stage j all become just one frame at stage j. However this still leaves many gauge U for which the new qubit string state representation is not gauge invariant.

Another method of arriving at a gauge invariant description of rational string states is based on the description of the kinematics of a quantum system by states in a Hilbert space $\mathcal{H}$, based on the $SU(2)$ group manifold. Details of this, generalized to all compact Lie groups, are given in [22] and [21]. In essence this extends the well known situation for angular momentum representations of states based on the manifold of the group $SO(3)$ to all compact Lie groups.

For the angular momentum case the the action of any rotation on the states $|l, m\rangle$ gives linear combinations of states with different m values but with the same l value. The Hilbert space spanned by all angular momentum eigenstates can be expanded as a direct sum

$$\mathcal{H} = \bigoplus_l \mathcal{H}_l \tag{30}$$

where $l = 0, 1, 2, \cdots$ labels the irreducible representations of $SO(3)$. Qubits can be associated with this representation by choosing two different l values, say l_0 and l_1. Then any states in the subspaces $\mathcal{H}_{l_0}$ and $\mathcal{H}_{l_1}$ correspond to the $|0\rangle$ and $|1\rangle$ qubit states respectively. These states are invariant under all rotations. Extension of this construction to all finite qubit strings gives a representation of natural numbers, integers and rational numbers that is invariant under all $SO(3)$ gauge transformations.

This development can also be carried out for any compact Lie group where the quantum kinematics of a system is based on the group manifold [22, 21]. In the case of $SU(2)$ Eq. 30 holds with $l = 0, 1/2, 1.3/2, \cdots$. The momentum

subspaces $\mathcal{H}_l$ are invariant under all $SU(2)$ transformations. As in the angular momentum case one can use this to describe states of qubits as

$$\begin{aligned} |0\rangle &\to \mathcal{H}_{l_0} \\ |1\rangle &\to \mathcal{H}_{l_0} \end{aligned} \tag{31}$$

that are $SU(2)$ invariant.

This construction can be extended to states of finite strings of qubits. Details of the mathematics needed for this, applied to graphs on a compact space manifold, are given in [21]. In this way one can describe representations of rational string numbers that are $SU(2)$ gauge invariant for all gauge U.

There is a possible connection between these representations of numbers and the Ashtekar approach to loop quantum gravity. The Ashtekar approach [21] describes G valued connections on graphs defined on a $3D$ space manifold where G is a compact Lie group. The Hilbert space of states on all graphs can be represented as a direct sum of spaces for each graph. The space for each graph can be further decomposed into a sum of gauge invariant subspaces. This is similar to the spin network decomposition first described by Penrose [23].

The connection to qubit strings is made by noting that strings correspond to simple one dimensional graphs. States of qubit strings are defined as above by choosing two l values for the space of invariant subspaces as in Eq. 31. It is hoped to describe more details of this connection in future work.

Implementation of this approach to reduction of the frame field one still leaves a one dimensional line of iterated frames. The line is finite, Fig. 2, one way infinite, Fig. 3, two way infinite, Fig. 4, or closed, Fig. 5. Because the two way infinite and cyclic fields have no abstract external sets of real and complex numbers and no abstract external space time, it seems appropriate to limit consideration to them. Here the cyclic field may be the most interesting because the number of the iterations in the cycle is undetermined. If it were possible to reduce the number to 0, then one would end up with a picture like Fig. 1 except that the R and C base of the frame would be identified in some sense with the gauge invariant representations described in the frame. Whether this is possible, or even desirable, or not, is a problem left to future work.

Another approach to connecting the frame field to physics is based on noting that the independence of physical theories from the properties of different real and complex number representations can be expressed using notions of symmetry and invariance. This is that

> *All physical theories to date are invariant*
> *under all $SU(2)$ gauge transformations of the qubit based*
> *representations of real and complex numbers.*

Note that the gauge transformations apply not only to the qubit string states but also to the arithmetic relations and operations on the string states, Eqs. 20 and 21, to sequences of states, to the Cauchy condition Eq. 23, and to the definition of the basic field operations on the real numbers.

The symmetry of physical theories under these gauge transformations suggests that it may be useful to drop the invariance and consider at the outset

candidate theories that break this symmetry. These would be theories in which some basic aspect of physics is representation dependent. One approach might be to look for some type of action whose minimization, coupled with the above requirement of gauge invariance, leads to some restriction on candidate theories. This gauge theory approach is yet another aspect to investigate in the future.

7   Discussion

There are some points of the material presented here that should be noted. The gauge transformations described here apply to finite strings of qubits and their states. These are the basic objects. Since these can be used to represent natural numbers, integers, and rational numbers in quantum mechanics, one can, for each type of number, describe different representations related by $SU(2)$ gauge transformations on the qubit string states. Here this description was extended to sequences of qubit string states that satisfied the Cauchy condition to give different representations of the real numbers.

A reference frame was associated to each real and complex number representation. Each frame contains a representation of all physical theories as mathematical structures based on the real and complex number representation base of the frame. If the space time manifold is considered to be a $4 - tuple$ of the real numbers, then each frame includes a representation of space time as a $4 - tuple$ of the real number representation.

If one takes this view regarding space time, it follows that for all frames with an ancestor frame, an observer outside the frame field or an observer in an ancestor frame sees that the points of space time in each descendent frame have structure as each point is an equivalence class of Cauchy sequences of (or a Cauchy operator on) states of qubit strings. It is somewhat disconcerting to regard space time points as having structure. However this structure is seen only by the observers noted above. An observer in a frame F does not see his or her own space time points as having structure because the real numbers that are the base of his frame do not have structure. Relative to an observer in frame F, the space time base of the frame is a manifold of featureless points.

It should also be noted that even if one takes the view that the space time manifold is some noncompact, smooth manifold that is independent of R^4, one still has the fact that functions from the manifold to the real numbers are frame dependent in that the range set of the functions is the real number representation base of the frame. Space time metrics are good examples of this. As is well known they play an essential role in physics.

In quite general terms, this work is motivated by the need to develop a coherent theory that treats mathematics and physics together as a coherent whole [24]. It may be hoped that the approach taken here that describes fields of frames based on different representations of real and complex numbers will shed light on such a theory. The point that these representations are based on different representations of states of qubit strings shows the basic importance of these strings to this endeavor.

Finally it should be noted that the structure of frames emanating from frames has nothing to do with the Everett Wheeler view of multiple universes [25]. If these multiple universes exist, then they would exist within each frame in the field.

Acknowledgements

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